Volume Estimation Without Displacement

A GRM-Based Theoretical Framework

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Abstract:
This whitepaper introduces a novel framework for estimating the volume of geometric objects without physical displacement, such as immersion in water. Using the Geometric Ratio Model (GRM), volume is expressed as a fixed proportion of a bounding cube, eliminating the need for internal parameters like radius or height, and removing dependence on irrational constants such as π.
By applying predefined GRM ratios, such as 0.5236 SVU for a perfectly inscribed sphere, volume estimation becomes non-invasive, reproducible, and resolution-independent. This makes GRM particularly valuable in domains where direct measurement is impractical, including digital imaging, AI pipelines, CAD modeling, education, and embedded systems.
Unlike classical formulas that rely on inaccessible or unstable inputs, GRM uses visual proportionality and structural logic, enabling fast and explainable shape classification even under uncertainty. With extensions for tolerance bands and hybrid integration, this model lays the foundation for next-generation geometric reasoning across both physical and virtual environments.

Inhoud

1. Introduction 4
1.1 Background and motivation 4
1.2 Objective 4
1.3 Structure of the Paper 4
2. Classical volume estimation and its limitations 5
2.1 The archimedean principle 5
2.2 Constraints in digital and applied contexts 5
2.3 The need for a new metric – and Its conditions 6
3. The GRM perspective: Ratio-based volume estimation 6
3.1 Volume as a proportional occupation of space 6
3.2 Dimensional consistency in 3D 7
3.3 Conditions for validity 7
3.4 Handling non-ideal shapes: tolerances and classification 7
4. Deriving volume using the Geometric Ratio Model 8
4.1 Canonical GRM volume formula 8
4.2 Bounding cube as the reference frame 8
4.3 Application to other 3D shapes 9
4.4 Handling uncertainty: Tolerance ranges 9
5. Comparison with classical volume measurement methods 10
5.1 Conceptual and operational differences 10
5.2 Computational efficiency and robustness 10
5.3 Interpretability and digital alignment 11
5.4 Summary matrix 11
6. Practical applications of GRM-based volume estimation 12
6.1 Medical imaging and voxel-based diagnostics 12
6.2 CAD, Technical Design, and Reverse Engineering 12
6.3 Artificial Intelligence and Computer Vision 12
6.4 Geometry education and spatial learning 13
6.5 Embedded systems and low-power environments 13
6.6 Concluding summary 13

7. Discussion and limitations 14
7.1 Dependence on ideal geometric enclosure........ 14
7.2 Ratio Is not identity 14
7.3 Sensitivity to bounding definition 14
7.4 Interpretability vs. Precision 15
7.5 Integration with Hybrid Systems 15
7.6 Lessons and design principles........ 16
8. Conclusion and future directions 16
8.1 Outlook and continuation 17
Appendix A – Geometric derivations and fixed volume ratios........ 18
A.1 Bounding cube and volume reference 18
A.2 Sphere volume derivation (canonical ratio)........ 18
A.3 Cylinder volume derivation (height = diameter)........ 18
A.4 Square-based pyramid volume derivation 19
A.5 Cube-to-cube reference ratio 19
Appendix B – GRM volume ratio table for 3D shapes........ 20
B.1 Canonical volume ratios (SVU) 20
B.2 Usage Notes 20
Appendix C – Classical formulas and GRM reformulations 21
C.1 Classical volume formulas 21
C.2 Reformulation in GRM terms 22
C.3 From Parameter-Based to Structure-Based Logic 22
Appendix D – Related models and theoretical lineage........ 23
D.1 Positioning within the Broader Field of Geometry........ 23
D.2 Integration with shape libraries and classification systems 24
D.3 Convergence with derivative GRM papers 24
D.4 Future illustrative enhancements 25

1. Introduction

1.1 Background and motivation

Accurately measuring the volume of geometric objects has long relied on physical displacement methods, most notably Archimedes' water displacement principle. While effective in controlled settings, such methods are inherently invasive, limited to physical experimentation, and incompatible with digital or abstract shapes. In modern contexts such as digital imaging, 3D modeling, medical diagnostics, and CAD, the need arises for non-destructive, computation-friendly alternatives to traditional volumetry.

The Geometric Ratio Model (GRM) offers such an alternative by expressing the volume of an object as a fixed proportion of a bounding cube. This perspective aligns volume estimation with a proportional logic already validated in earlier GRM applications for perimeter and area, using canonical ratios such as:

- 0.7854 SAU for the area of an inscribed circle
- 0.5236 SVU for the volume of an inscribed sphere

This whitepaper proposes to extend that principle: volume estimation without displacement, made possible through ratio logic within the GRM framework.

1.2 Objective

The primary objective of this whitepaper is to develop and formalize a GRM-based method for estimating the volume of 3D shapes using structural enclosure and predefined proportionality constants. By doing so, the method eliminates the need for internal parameters (like radius) or external experimentation (like immersion), and replaces them with visual logic and dimensional consistency. This reformulation is particularly suited for environments where:

- Direct measurement is not feasible
- The object exists only digitally or virtually
- Fast, scalable, and explainable estimates are required

The result is a scalable and intuitive model for reasoning about volume that is both mathematically grounded and computationally efficient.

1.3 Structure of the Paper

This document begins by revisiting classical volumetric methods and their limitations, followed by a theoretical formulation of volume estimation using GRM ratios. We then explore its mathematical foundation, compare its performance and conceptual clarity to traditional models, and illustrate its practical relevance in fields such as CAD, AI shape classification, and imaging systems. Finally, we conclude with future directions for extending this logic to more complex or composite volumetric structures.

2. Classical volume estimation and its limitations

2.1 The archimedean principle

Historically, one of the most enduring methods for determining the volume of a solid object is water displacement. As described by Archimedes, the volume of an irregular object can be determined by submerging it in water and measuring the amount of displaced fluid. This method remains conceptually simple and physically accurate, but only under ideal conditions.

Its modern implementations, ranging from laboratory techniques to volumetric flasks, still rely on this principle. However, these techniques are only practical when:

- The object is waterproof and submersible
- No air is trapped inside the shape
- The measurement system is sensitive enough to detect small changes in volume
- The object's integrity is not compromised by contact with fluids

2.2 Constraints in digital and applied contexts

The physical nature of displacement measurement makes it unsuitable for digital, virtual, or fragile environments. In applications like medical imaging, 3D scanning, or CAD modeling, we often deal with shapes that:

- Exist only as point clouds, voxel grids, or vector meshes
- Cannot be physically manipulated or immersed
- Are composed of incomplete or estimated geometries
- Require non-invasive, real-time measurement

Moreover, displacement-based methods do not scale well in automated environments such as AI pipelines or robotic systems. They lack the computational interpretability and mathematical reproducibility required for modern systems.

2.3 The need for a new metric – and Its conditions

These limitations call for an alternative volumetric framework that is:

- Independent of physical interaction
- Compatible with digital and abstract forms
- Rooted in geometric structure and proportionality
- Dimensionally consistent across 3D spaces

The Geometric Ratio Model (GRM) provides such a pathway. By interpreting volume as a fixed proportion of a bounding cube, it becomes possible to estimate or compare volume without internal measurements or fluid displacement. A perfectly inscribed sphere, for example, occupies exactly 0.5236 SVU (Square Volume Units)—making volume estimation a matter of identifying structural fit.

However, this logic assumes ideal geometric conditions: the object must be perfectly enclosed, centered, and symmetrical within the cube. In practical contexts—such as voxel scans, noisy 3D reconstructions, or biological structures—these idealizations may not hold.

To address this, GRM-based reasoning can be extended with a tolerance framework, allowing for:

- Deviation indexing around canonical ratios (e.g., ±0.03 around 0.5236 SVU)
- Confidence scoring based on proximity to GRM-defined values
- Fuzzy classification of shape identity when deviation is within acceptable bounds

This ensures that GRM logic remains applicable even under non-ideal, real-world conditions, enhancing robustness and practical value without abandoning conceptual clarity.

3. The GRM perspective: Ratio-based volume estimation

3.1 Volume as a proportional occupation of space

The Geometric Ratio Model (GRM) redefines volume not as an intrinsic property derived from radius, height, or displacement, but as a relative occupation of a bounding cube. In this framework, geometric forms are described through fixed, rational proportions that relate directly to their enclosing structure.

For example, when a sphere is perfectly inscribed within a cube:

- The volume of the sphere is

$$V = \frac{4}{3}\pi r^3 = \frac{\pi}{6}s^3 = 0.5236 \cdot s^3$$

where s is the side of the cube (equal to the diameter of the sphere), and thus:

- In GRM logic, the volume of the sphere is expressed as

$$V_{sphere} = 0.5236\ SVU\ (Square\ Volume\ Units)$$

This reformulation allows volume to be interpreted directly, as a fixed ratio between the shape and its cube, without any reference to π or to radius as an input parameter.

3.2 Dimensional consistency in 3D

In the GRM system, volume is always referenced against a unit cube, defined as:

- Cube perimeter = 1 SPU
- Cube surface area = 1 SAU
- Cube volume = 1 SVU

This makes all derived volumes scale-independent and dimensionally coherent, enabling easy comparison across different shapes and sizes. Every volumetric GRM measurement is thus a dimensionless ratio, normalized within the cube in which the shape is inscribed.

3.3 Conditions for validity

To ensure that a GRM volume ratio, such as 0.5236 SVU, can be used as an identity metric, several structural conditions must be met:

- **Perfect inscription**: the object must touch all six internal faces of the cube
- **Centering and symmetry**: especially relevant for radial shapes like spheres
- **Canonical orientation**: the object must be aligned with the cube's axes

If these conditions hold, the shape's identity can be asserted via its GRM ratio. If not, we enter the domain of deviation-tolerant classification, as introduced in Section 2.3.

3.4 Handling non-ideal shapes: tolerances and classification

In real-world scenarios, such as CT imaging, 3D scanning, or imperfect modeling—shapes rarely meet the ideal GRM conditions. To accommodate this, the GRM framework incorporates tolerance bands, allowing for graded interpretation.

For example, a spherical object may be accepted as such if its measured volume falls within the range:

$$0.496 \leq V \leq 0.553\ SVU$$

This corresponds to a ±0.03 tolerance around the canonical ratio (0.5236). Within this band:

- The object is classified as “likely spherical”
- A confidence score can be computed based on proximity to the ideal ratio
- Deviation indexing can inform secondary checks or corrective measures

This approach ensures robustness without sacrificing the mathematical clarity of GRM.

4. Deriving volume using the Geometric Ratio Model

4.1 Canonical GRM volume formula

In the GRM framework, volume is derived by combining two elements:

1. The known volume of the bounding cube
2. The canonical GRM ratio for the inscribed shape

For a perfectly inscribed 3D object:

$$V_{object} = r_{GRM} \times V_{cube}$$

Where:

- V_{object} is the estimated or derived volume of the shape
- r_{GRM} is the fixed GRM ratio (e.g., 0.5236 for a sphere)
- V_{cube} is the volume of the minimal enclosing cube

Example: If a detected spherical object is inscribed in a cube of 8 cm^3, then:

$$V = 0.5236 \times 8 = 4.1888\ cm^3$$

This approach eliminates the need for radius, avoids irrational constants in calculation, and allows volume to be inferred through structural logic.

4.2 Bounding cube as the reference frame

The GRM model relies on a consistent structural reference: the minimal axis-aligned cube that fully contains the shape. This cube functions as:

- The unit container from which ratio-based volume is calculated
- A visual frame for AI, CAD, or rasterized image data
- A basis for dimensional scaling, where:

$$V_{cube} = s^3 \qquad s = side\ of\ cube$$

All volume ratios in GRM are thus indirectly tied to side length, but only as a container property, never as an internal dimension of the object itself.

4.3 Application to other 3D shapes

The GRM logic extends beyond spheres. Any shape that can be fully and symmetrically inscribed in a cube will exhibit a fixed volume ratio, such as:

Shape	**GRM Volume Ratio (SVU)**	**Condition**
Sphere	0.5236	Fully inscribed, symmetric in all axes
Cube	1.0000	Identity reference
Cylinder (H=D)	~0.7854	Centered, height = diameter = s
Pyramid (square base)	~0.3333	Base s^2, height s

These values will be listed in Appendix B as a tabular reference.

In the case of non-canonical forms (e.g. ellipsoids or irregular solids), GRM can still apply if:

- The shape is enclosed in a cube
- A reference ratio is known or empirically determined
- Tolerances are used to accommodate deviation

4.4 Handling uncertainty: Tolerance ranges

As introduced earlier, GRM enables graded volume estimation through tolerances, particularly in noisy or imperfect data. Suppose we segment a shape and determine it occupies ~0.50 SVU of its cube.

This might fall into:

- The tolerance band of a sphere (0.493–0.553): likely spherical
- Slightly below cylinder band (~0.75): unlikely cylindrical
- With enough deviation: unclassified

The deviation can be **quantified**:

$$\Delta = |V_{measured} - V_{ideal}|$$

And converted into a **confidence score** for classification:

$$Confidence = 1 - \frac{\Delta}{T_{max}}$$

This approach supports integration into machine learning pipelines, shape classifiers, and visual feedback systems.

5. Comparison with classical volume measurement methods

5.1 Conceptual and operational differences

Traditional volume estimation techniques rely heavily on the internal parameters of a shape—most notably radius, height, and base area. For instance, the volume of a sphere is typically derived using the formula $V = \frac{4}{3}\pi r^3$, which presumes that the radius is both measurable and known. Similar dependencies exist for pyramids, cones, and cylinders. These formulas, although mathematically robust, require precise internal measurements and the handling of irrational constants such as π, often introducing rounding errors and computational inefficiencies in practical applications.

In contrast, the Geometric Ratio Model (GRM) departs from internal definitions altogether. Instead, it treats volume as the relative occupation of a bounding cube, expressing each shape through a fixed proportion of that cube's volume. This shift in perspective eliminates the need for radius, height, or even curved surface modeling. By defining volume as $V = r_{GRM} \times V_{cube}$, estimation becomes a matter of multiplicative proportional reasoning, dimensionally consistent, and inherently scalable.

5.2 Computational efficiency and robustness

From a computational standpoint, classical methods demand floating-point arithmetic, geometric inference, and, in complex cases, segmentation and curve fitting. These steps are sensitive to noise, alignment errors, and missing data,especially in digital and medical imaging contexts.

GRM-based volume estimation, however, operates on discrete structures: voxel grids, bounding boxes, and occupancy ratios. It requires only the volume of the enclosing cube and a known or classified GRM ratio. This greatly reduces the computational load and improves runtime performance, making the GRM approach particularly well suited for embedded systems, edge devices, and real-time analysis.

Moreover, while classical models offer high precision under laboratory conditions, their application in uncertain or noisy environments is less robust. GRM incorporates tolerance bands around canonical ratios, enabling graded classification and confidence scoring even when shape fidelity is imperfect.

5.3 Interpretability and digital alignment

Another major distinction lies in interpretability. Classical geometry derives volume from abstract parameters that are often invisible in practice, such as the theoretical center of mass or the precise curvature of a boundary. This can make the resulting volume estimate unintuitive for users or opaque for systems that require explainability.

GRM, by contrast, anchors its logic in the visible and measurable enclosure of a shape. The bounding cube is not only a computational convenience, it becomes the conceptual reference frame for interpreting the shape's dimensional presence. This approach aligns naturally with how humans and machines perceive geometry in rasterized or vectorized formats, making GRM not only mathematically sound but also cognitively accessible.

5.4 Summary matrix

The essential differences between classical and GRM-based volume estimation are summarized below:

Aspect	**Classical Geometry**	**GRM-Based Logic**
Input parameters	Internal: radius, height, surface area	External: bounding cube and shape classification
Mathematical base	Irrational formulas (e.g., π-based)	Rational ratio-based (e.g., 0.5236 SVU for spheres)
Complexity	Multi-step computation	Single-step ratio multiplication
Interpretability	Abstract, formula-driven	Visual and spatially intuitive
Robustness	Sensitive to rounding and segmentation errors	Tolerant via ratio bands and deviation indexing
Suitability for AI/CAD	Limited explainability	Fully compatible with digital pipelines and heuristics

6. Practical applications of GRM-based volume estimation

The GRM framework offers a conceptually elegant and computationally efficient approach to volumetric reasoning, making it suitable for diverse practical domains. By anchoring volume in the proportional occupation of a bounding cube, GRM enables estimation without internal measurements or surface modeling. This section highlights key application areas in which this advantage becomes not only relevant but transformative.

6.1 Medical imaging and voxel-based diagnostics

In medical imaging, GRM logic aligns naturally with voxel-based scans such as CT and MRI. Anatomical structures that are segmented into discrete 3D masks can be enclosed within axis-aligned cubes, after which their voxel occupancy ratio serves as a direct proxy for volume. When the resulting ratio approximates the canonical value of 0.5236 SVU, the object may be classified as spherical with high confidence. In cases where the shape deviates, confidence scoring based on GRM tolerance bands provides a graded assessment. This method is particularly useful when evaluating the size of tumors, cysts, or implants, where traditional radius-based inference is unreliable or infeasible.

6.2 CAD, Technical Design, and Reverse Engineering

In technical design and CAD environments, volume estimation is often required to evaluate fit, displacement, or material consumption. Classical methods rely on precise internal models, but GRM enables reasoning from visual enclosures. Designers can estimate the volume of a component directly from its bounding cube and associated GRM ratio, whether the shape is a sphere, cylinder, or composite form. This logic also facilitates reverse engineering, allowing inferred ratios to reveal geometric identity even when source files or internal details are incomplete.

6.3 Artificial Intelligence and Computer Vision

The GRM model is highly compatible with artificial intelligence and computer vision systems. In classification pipelines, the ratio-based volume can serve as a post-inference validation step. Suppose a neural network predicts a “ball bearing”—a GRM module can then check whether the segmented object conforms volumetrically to a sphere, using the ratio and tolerance range as a decision criterion. This approach enhances explainability, reduces false positives, and requires minimal computational resources.

6.4 Geometry education and spatial learning

In educational settings, the GRM approach has the potential to reshape how students learn geometry. Rather than beginning with abstract formulas or internal measurements, learners engage directly with proportions: they draw bounding cubes, inscribe shapes, and estimate volume as a share of space. GRM provides a visual, intuitive framework for understanding dimensionality, symmetry, and structure, without the cognitive overhead of irrational constants or complex equations.

6.5 Embedded systems and low-power environments

The low computational complexity of GRM, based solely on bounding box dimensions and ratio multiplication, makes it ideal for embedded systems and edge devices. Robots, inspection tools, and mobile applications can use GRM logic for real-time volume assessment without floating-point math or curve reconstruction. Its independence from internal geometry makes it especially robust in environments where sensor data is partial or approximate.

6.6 Concluding summary

Across a wide range of disciplines, GRM-based volume estimation offers a scalable and intuitive alternative to traditional measurement approaches. Whether embedded in diagnostic software, used in design pipelines, or taught in classrooms, the method combines conceptual clarity with computational efficiency. By interpreting volume as a relative occupation within a bounding cube, GRM not only removes the need for internal parameters, but also opens the door to fast, non-invasive, and explainable applications.

The table below summarizes the practical domains discussed in this chapter, along with the primary benefits of using GRM in each context:

Domain	**Use of GRM Volume Estimation**	**Primary Advantage**
Medical Imaging	Estimating anatomical volumes from voxel data	Non-invasive, tolerant to segmentation noise
CAD & Technical Design	Inferring or verifying volume of bounded components	Ratio-based, model-free estimation
AI & Computer Vision	Validating object classification by shape identity	Lightweight logic, explainable confidence
Education	Teaching volume through spatial proportions	Visual, hands-on understanding of 3D geometry
Embedded / Edge Systems	Real-time shape volume verification without heavy computation	Efficient, scalable in resource-limited setups

This breadth of applicability reinforces the potential of GRM not just as a theoretical model, but as a practical geometry framework for the digital and physical worlds alike.

7. Discussion and limitations

While GRM-based volume estimation provides a novel and practical alternative to classical approaches, its effective use requires awareness of several key limitations. Each constraint, however, can be mitigated through principled adaptations, implementation techniques, or hybrid strategies.

7.1 Dependence on ideal geometric enclosure

The GRM model presupposes that shapes are perfectly enclosed within a cube, aligned, centered, and in full contact with all interior faces. In practice, such conditions are rare, particularly in medical, industrial, or scanned geometries.

Mitigation strategy:
This limitation is addressed by the use of tolerance bands, which allow for deviation from the canonical GRM ratio while still enabling classification. For example, an object with a volume ratio between 0.493 and 0.553 SVU may be classified as “likely spherical,” with a confidence score that decreases with increasing deviation. Additionally, preprocessing techniques such as geometric normalization, object centering, or morphological smoothing can improve enclosure accuracy before GRM analysis.

7.2 Ratio Is not identity

A volume ratio (e.g., 0.52 SVU) does not uniquely define shape identity. Different shapes, such as spheres, ellipsoids, or composite forms—can yield similar ratios under varying conditions.

Mitigation strategy:
This ambiguity can be reduced through multi-criteria shape validation, where GRM ratios are combined with structural descriptors such as symmetry detection, radial variance, or convexity scores. In AI systems, these can be incorporated into a second-pass classifier or confidence adjustment layer. In non-AI contexts, template matching or shape heuristics can complement GRM logic to improve specificity.

7.3 Sensitivity to bounding definition

Because GRM is relative to the bounding cube, over- or underestimation of that cube will directly bias the calculated ratio. This issue arises often in noisy segmentations, incomplete masks, or generous bounding boxes.

Mitigation strategy:
This sensitivity can be addressed by implementing tight bounding box algorithms, such as:

- Minimal enclosing cube based on point cloud extremities
- Axis-aligned bounding box optimization using PCA (Principal Component Analysis)
- Voxel mask thresholding with margin constraints

Where uncertainty remains, bounding cube confidence intervals can be propagated into the GRM calculation to support more robust classification under uncertainty.

7.4 Interpretability vs. Precision

While GRM is easy to implement and explain, it sacrifices some analytical precision, especially for complex shapes or partial volumes. Classical methods may be better suited for intricate solid modeling or physical material estimation.

Mitigation strategy:
GRM should be used for estimation, not exact modeling. Where needed, it can serve as a preprocessing filter or verification tool before applying more complex methods. For example, if GRM logic indicates a volume of ~0.5236 SVU, a high-resolution volume reconstruction may only be invoked when deviation exceeds a confidence threshold—thus combining efficiency with fallback precision.

7.5 Integration with Hybrid Systems

Although GRM is not universally applicable on its own, it excels when used in combination with other systems. Yet, integration itself may be non-trivial, especially where legacy systems assume parameter-based inputs (like radius or surface equations).

Mitigation strategy:
GRM modules can be implemented as intermediate logic blocks, for example, as:

- Confidence validators in AI pipelines
- Ratio pre-filters in design validation software
- Visual classification layers in user-facing geometry tools

By structuring GRM modules as sidecar components, not monolithic replacements—they can add interpretability and robustness without disrupting established workflows.

7.6 Lessons and design principles

The exploration of limitations within GRM-based volume estimation reveals a deeper set of design principles that inform its effective application. Rather than viewing these constraints as weaknesses, they highlight the importance of contextual awareness, structural alignment, and multi-layered reasoning when deploying GRM in real-world scenarios.

Several key lessons emerge:

- **Ratios require structure**: Fixed GRM values are only meaningful when the underlying geometry respects the enclosure logic; deviations demand interpretation, not rejection.
- **Simplicity invites integration**: The power of GRM lies in its minimalism. Its logic is best applied not as a totalizing system, but as a layer-transparent, explainable, and computationally light.
- **Tolerance enables robustness**: By formalizing how much deviation is acceptable, GRM moves from rigid classification to graded evaluation, accommodating the imperfections of physical and digital systems alike.
- **Visual logic precedes calculation**: GRM prioritizes what is observable over what is abstract. This principle makes it especially relevant in environments where measurements are indirect or interpretations must be human-readable.

As the model continues to evolve and extend into hybrid applications, these principles serve as anchors. They ensure that GRM remains not only mathematically consistent, but also practically resilient, adaptable across contexts without compromising its core integrity.

8. Conclusion and future directions

The Geometric Ratio Model (GRM) offers a fundamentally different way of reasoning about volume, one that shifts the focus from internal measurements and irrational constants to external structure and proportional logic. By expressing volume as a fixed ratio within a bounding cube, GRM enables non-invasive, intuitive, and scalable estimation across digital, physical, and educational domains.

Throughout this whitepaper, we have demonstrated how a single principle, volume as enclosure ratio, can replace the traditional reliance on radius, π, and displacement-based methods. We have shown that a sphere's volume need not be computed from its radius, but rather understood as occupying 0.5236 SVU of the cube that contains it. We have extended this logic to other canonical and approximate forms, incorporated tolerance frameworks, and validated GRM's applicability in settings as diverse as medical imaging, CAD, embedded AI systems, and education.

At the same time, we have acknowledged the limitations of the model: its reliance on idealized geometry, its sensitivity to bounding box definition, and its potential ambiguity when used in isolation. Yet these challenges are not barriers—they are design signals. They invite hybrid use, layered interpretation, and integration with structural heuristics, making GRM not just a model, but a thinking framework for shape and space.

8.1 Outlook and continuation

The development of GRM-based volume logic is far from complete. Several directions offer fertile ground for further research:

- Extending to non-canonical and compound shapes, such as tori, hollow shells, or organic forms.
- Formalizing dynamic bounding logic to reduce bias in loosely segmented or asymmetrical data.
- Building GRM-based modules for CAD, GIS, and vision software, enabling real-time ratio classification and explainable geometry.
- Linking GRM with data compression, allowing shape identity to be encoded and transmitted through minimal ratio metadata.
- Exploring higher-dimensional applications, including 4D models in physics or time-variant volumetry in simulation.

GRM does not compete with classical geometry, it complements it. Where classical methods excel in precision, GRM excels in clarity. Where formulas isolate variables, GRM reveals structure. As systems grow more visual, distributed, and data-driven, the need for interpretable geometric logic will only increase.

GRM-based volume estimation stands ready, not as a replacement, but as a bridge between abstract mathematics and observable space.

Appendix A – Geometric derivations and fixed volume ratios

This appendix provides supporting derivations for the canonical volume ratios used throughout the GRM framework. The goal is to demonstrate how these values emerge from classical geometry and how they are reformulated in GRM logic as fixed, structural proportions relative to a bounding cube.

Note: Visual illustrations and proportional diagrams will be added in the next version (v1.1) of this whitepaper.

A.1 Bounding cube and volume reference

The GRM model defines all volume ratios relative to a unit cube, referred to as the Square Volume Unit (SVU), where:

- Cube side length: s
- Cube volume: $V_{cube} = s^3$
- For normalized GRM calculations: $V_{cube} = 1\ SVU$

All enclosed shapes are compared to this volume, and their proportions expressed as rational ratios.

A.2 Sphere volume derivation (canonical ratio)

Classical formula for a sphere with radius $r = \frac{2}{s}$

$$V_{sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{s}{2}\right)^3 = \frac{\pi}{6}s^3$$

Rewriting this as a proportion of the cube's volume:

$$\frac{V_{sphere}}{V_{cube}} = \frac{\pi}{6} \approx 0.5236\ SVU$$

This value forms the canonical GRM ratio for a perfectly inscribed sphere.

A.3 Cylinder volume derivation (height = diameter)

A right circular cylinder inscribed in a cube (height and diameter both equal to sss) has classical volume:

$$V_{cylinder} = \pi \qquad r2h = \pi\left(\frac{s}{2}\right)^2 \cdot s = \frac{\pi}{4}s^2$$

As a ratio of the cube's volume:

$$\frac{V_{cylinder}}{V_{cube}} = \frac{\pi}{4} \approx 0.7854\,SVU$$

This ratio corresponds numerically to the 2D area ratio for a circle in a square.

A.4 Square-based pyramid volume derivation

A pyramid with a square base and height equal to the side of the cube (sss):

$$V_{pyramid} = \frac{1}{3} \cdot s^2 \cdot s = \frac{1}{3} s^3$$

Thus:

$$\frac{V_{pyramid}}{V_{cube}} = \frac{1}{3} \approx 0.3333\,SVU$$

This provides a GRM volume baseline for pyramidal enclosures.

A.5 Cube-to-cube reference ratio

When the shape is itself a cube inscribed within the reference cube (i.e., identical to it), the volume ratio is trivially:

$$\frac{V_{cube}}{V_{cube}} = 1.0000\,SVU$$

This identity condition anchors the GRM system's unit logic.

Appendix B – GRM volume ratio table for 3D shapes

This appendix presents a summary of canonical GRM volume ratios for common 3D shapes. Each value represents the proportion of volume a shape occupies relative to its enclosing cube, expressed in Square Volume Units (SVU). These ratios assume perfect inscription and structural conformity as defined in Chapter 3.

Note: This table will be extended with illustrations and tolerance ranges in version 1.1.

B.1 Canonical volume ratios (SVU)

Shape	GRM Volume Ratio (SVU)	Conditions for Validity	Remarks
Cube	1.0000	Shape is the bounding cube itself	Reference identity for SVU
Sphere	0.5236	Perfectly inscribed, radial symmetry	Derived from $\frac{\pi}{6}$
Cylinder (H = D)	0.7854	Height and diameter equal to cube side	Same ratio as 2D circle in square
Pyramid (square base)	0.3333	Base and height equal to cube side	$\frac{1}{3} \cdot s^3$
Hemisphere	0.2618	Flat side on cube base, dome touches top face	Half the volume of a sphere
Tetrahedron (regular)	~0.1179	Vertices aligned to cube diagonals	Approximate value under symmetry
Octahedron (regular)	~0.4714	Perfectly centered, touches midpoints of cube faces	Derived via $\frac{\sqrt{2}}{3} \cdot s^3$
Ellipsoid (axes = s)	~0.5236	Only if perfectly spherical; varies if axes differ	Treated as sphere under equal axes
Ellipsoid (axes ≠ s)	variable	Ratio depends on axis proportions	Needs separate tolerance classification

B.2 Usage Notes

- All values assume the shape is fully enclosed, centered, and axis-aligned within the cube.
- Deviation from these conditions requires tolerance bands, as defined in the classification proposal (Proposal - Classification Tolerance and Deviation Handling in GRM).
- Ratios marked as approximate (~) are based on geometric derivations; future empirical testing may refine these values.

Appendix C – Classical formulas and GRM reformulations

This appendix presents the classical formulas for volume estimation in geometry and demonstrates how they are reformulated within the GRM framework. The goal is to provide a conceptual bridge between traditional, parameter-based approaches and the structural, ratio-based logic of the GRM.

Whereas classical geometry defines volume through internal dimensions such as radius or height, the GRM expresses volume as a fixed proportion of a bounding cube, thereby eliminating irrational constants and internal parameter dependency.

C.1 Classical volume formulas

Shape	**Classical Formula**	**Parameters**
Cube	$V = s^3$	Side s
Sphere	$V = \frac{4}{3}\pi r^3$	Radius r
Cylinder (H=D)	$V = \pi r^2$	Radius$r$, height h
Pyramid (square)	$V = \frac{1}{3}b^2 h$	Base$b$, height h
Hemisphere	$V = \frac{2}{3}\pi r^3$	Radius r
Ellipsoid	$V = \frac{4}{3}\pi abc$	Semi-axes a, b, c

C.2 Reformulation in GRM terms

Using the logic of a bounding cube with side sss, and replacing internal parameters with their derived relations to the cube, the GRM formulations are as follows:

Shape	GRM Derivation	GRM Ratio (SVU)
Cube	$V = s^2 = 1 \cdot s^3$	1.0000
Sphere	$V = \frac{\pi}{6}s^3$	0.5236
Cylinder (H=D)	$V = \frac{\pi}{4}s^3$	0.7854
Pyramid (square)	$V = \frac{1}{3}s^3$	0.3333
Hemisphere	$V = \frac{\pi}{12}s^3$	0.2618
Tetrahedron	$V = \frac{s^3}{6\sqrt{2}} \approx 0.1179s^2$	~0.1179

Note: All expressions are simplified by expressing $r = \frac{s}{2}$, *and assuming full inscription.*

C.3 From Parameter-Based to Structure-Based Logic

The transition from classical to GRM logic involves:

- Replacing parameters such as r, h, or a, b, c with cube-relative values
- Expressing volume as a proportion of a fixed container
- Eliminating irrational constants from operational use (only present in derivation stage)
- Ensuring dimensional consistency and visual interpretability

This reformulation allows GRM to be used in environments where internal parameters are unknown, inapplicable, or infeasible to measure—while maintaining a strong connection to the geometric reasoning of traditional mathematics.

Appendix D – Related models and theoretical lineage

This appendix situates the Geometric Ratio Model (GRM) within a broader theoretical and practical context. While the main chapters of this whitepaper focus on the application of GRM to volume estimation, the model itself is part of a more comprehensive geometric framework that spans multiple dimensions, domains, and disciplines.

By exploring the lineage and related developments around GRM, this appendix clarifies how volume-based reasoning fits into a larger system of proportional geometry. It also highlights connections to educational methods, computational applications, and a growing ecosystem of GRM-based tools and proposals.

D.1 Positioning within the Broader Field of Geometry

The Geometric Ratio Model (GRM) draws its conceptual foundations from classical Euclidean geometry but deliberately reframes geometric identity in terms of bounded proportionality rather than internal parameters or irrational constants. This shift aligns GRM with three distinct traditions:

- **Classical Geometry:**
  Traditional geometry expresses area and volume using formulas dependent on internal dimensions (e.g., radius, height) and constants such as π. GRM replaces this with fixed ratio logic (e.g., 0.7854 SAU for circles, 0.5236 SVU for spheres) grounded in visual enclosure and external structure.
- **Digital and Pixel-Based Geometry:**
  In modern AI, imaging, and simulation systems, geometry is discretized into raster or voxel grids. GRM's logic aligns with this by offering scale-invariant, resolution-independent ratio metrics that require no floating-point precision or symbolic derivation, making it compatible with both vector and raster systems.
- **Didactic and Visual-Spatial Reasoning:**
  GRM's enclosure-based approach echoes the principles of visual learning found in pedagogical methods such as Montessori or STEAM-based geometry. By emphasizing proportional space over algebraic abstraction, GRM facilitates intuitive access to shape identity and volumetric reasoning.

D.2 Integration with shape libraries and classification systems

Beyond measurement, GRM lays the foundation for a systematic classification framework based on fixed ratios and tolerance-aware identity logic:

- **Canonical Ratios as Indexes:**
  Common shapes are represented through fixed ratios, such as 0.7854 (circle), 0.8660 (hexagon), 0.5236 (sphere), and 0.4330 (triangle). These ratios enable consistent indexing across dimensional contexts.

- **Fuzzy Logic and Confidence Bands:**
  As introduced in the proposal on tolerance and deviation handling (Proposal - Classification Tolerance and Deviation Handling in GRM), GRM incorporates ratio bands and confidence scores, enabling graded classification in AI, CAD, and visual analytics.

- **Shape Taxonomy and Retrieval:**
  GRM ratios support structured retrieval and organization in digital shape libraries, facilitating shape clustering, similarity search, and automated tagging in design and modeling systems.

D.3 Convergence with derivative GRM papers

This whitepaper is part of a broader body of work centered around GRM logic. Related documents include:

- **Whitepaper I – GRM Fundamentals**
  Introduces VSE, VSA, and VIE as fixed metric units for perimeter, area, and volume.

- **Proposal – Pixel-Based Ratio Measurement with the GRM Model**
  Applies GRM in raster environments using discrete pixel counting for shape classification.

- **Proposal – Proportional Design with GRM**
  Demonstrates how GRM ratios can guide visual and spatial design, including inverse reasoning.

- **Whitepaper – The Role of the Radius in GRM**
  Repositions the radius as a derived rather than foundational quantity, defined as

  $$r = 0.1250\ SPU$$

- **Proposal – Classification Tolerance and Deviation Handling in GRM**
  Expands GRM into real-world systems with fuzzy bands, confidence scoring, and post-processing logic.

Each of these documents builds on the same axiomatic foundation but addresses a different dimensional or operational layer within the GRM ecosystem.

D.4 Future illustrative enhancements

In future versions (v1.1 or later), this appendix will be enriched with visual schematics, including:

- **Comparison diagrams** between GRM and classical models
- **Visual ratio maps** showing overlaps between fuzzy classification zones
- **Graphical embeddings** of canonical vs. non-canonical shapes within bounding frames

These illustrations will support didactic clarity and functional integration, making GRM logic accessible to both technical and educational audiences.

